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Multiparametric quantum $gl(2)$: Lie bialgebras, quantum R-matrices and non-relativistic limits

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Abstract. Multiparametric quantum deformations of $gl(2)$ are studied through a complete classification of $gl(2)$ Lie bialgebra structures. From these, the non-relativistic limit leading to harmonic oscillator Lie bialgebras is implemented by means of a contraction procedure. New quantum deformations of $gl(2)$, together with their associated quantum R-matrices, are obtained, and other known quantizations are recovered and classified. Several connections with integrable models are outlined.

1. Introduction

A two-parametric quantum deformation of $gl(2)$ has been proved in [1] to provide the quantum group symmetry of the spin-$\frac{1}{2}$ XXZ Heisenberg chain with twisted periodic boundary conditions [2, 3]. In this context, the central generator I of the $gl(2)$ algebra plays an essential role in the algebraic introduction of the twisted boundary terms of the spin Hamiltonian through a deformation induced from the exponential of the classical r-matrix $r = J_3 \wedge I$. This seems not to be an isolated example, since the general construction introduced in [4] establishes a correspondence between models with twisted boundary conditions (see references therein) and multiparametric Reshetikhin twists [5] in which the Cartan subalgebra is enlarged with a (cohomologically trivial) central generator.

From another different physical point of view, $gl(2)$ can be also considered as the natural relativistic analogue of the one-dimensional harmonic oscillator algebra [6]. The latter (which is a non-trivial central extension of the (1 + 1) Poincaré algebra) can be obtained from $gl(2)$ (which is a trivial central extension of $sl(2,\mathbb{R}) \equiv so(2,1)$) through a generalized Inönü–Wigner contraction, that can be interpreted as the algebraic transcription of the non-relativistic limit connecting both kinematics. The direct applicability of quantum algebras in the construction of completely integrable many-body systems through the formalism given in [7] (that precludes the use of any transfer matrix technique by making use directly of the Hopf algebra axioms) suggests that a systematic study of quantum $gl(2)$ algebras would be related to the definition of integrable systems consisting in long-range interacting relativistic oscillators (see [8] for the construction of non-relativistic oscillator chains). Finally, note also that a $gl(2)$ induced deformation of the Schrödinger algebra has been recently used to construct a discretized version of the (1 + 1) Schrödinger equation on a uniform time lattice [9].

Up to now, much attention has been paid to quantum $GL(2)$ groups and their classifications [10–16], but a fully general and explicit description of quantum $gl(2)$ algebras is still lacking, although partial results can be already found in the literature [17–24]. Such a systematic approach to quantum $gl(2)$ algebras is the aim of the present paper, and the underlying Lie bialgebra structures and classical r-matrices will be shown to contain all the essential information characterizing different quantizations. In section 2, $gl(2)$ Lie bialgebras are fully obtained and classified into two multiparametric and inequivalent families. Their contraction to the harmonic oscillator Lie bialgebras is performed in section 3 by introducing a multiparameter generalization of the Lie bialgebra contraction theory [25] that allows us to perform the non-relativistic limit. Among the quantum deformations of the harmonic oscillator algebra whose Lie bialgebras are obtained, we find the one introduced in [26] in the context of link invariants. Finally, an extensive study of the quantizations of $gl(2)$ Lie bialgebras is given in section 4. New quantum algebras, deformed Casimir operators and quantum R-matrices are obtained and known results scattered through the literature are easily derived from the classification presented here. In particular, the quantum algebra corresponding to the quantum group $GL_{h,q}(2)$ [16] is constructed (recall that this is a natural superposition of both standard and non-standard deformations). Quantum symmetry algebras of the twisted XXZ and XXX models are identified, and it is shown how new quantum $gl(2)$ invariant spin chains can be systematically obtained from the new multiparametric deformations that have been introduced.

2. The $gl(2)$ Lie bialgebras

A Lie bialgebra (g, δ) is a Lie algebra g endowed with a map $\delta : g \to g\otimes g$ (the cocommutator) that fulfils two conditions: (i) δ is a 1-cocycle, i.e.

$$\delta([X, Y]) = [\delta(X),\ 1 \otimes Y + Y \otimes 1] + [1 \otimes X + X \otimes 1,\ \delta(Y)] \qquad \forall\, X, Y \in g. \tag{2.1}$$

(ii) The dual map $\delta^* : g^* \otimes g^* \to g^*$ is a Lie bracket on g^*.

A Lie bialgebra (g, δ) is called a coboundary Lie bialgebra if there exists an element $r \in g \wedge g$ (the classical r-matrix), such that

$$\delta(X) = [1 \otimes X + X \otimes 1,\ r] \qquad \forall\, X \in g. \tag{2.2}$$

When the r-matrix is a skewsymmetric solution of the classical Yang–Baxter equation (YBE) we shall say that $(g, \delta(r))$ is a *non-standard* (or triangular) Lie bialgebra, while when it is a skewsymmetric solution of the modified classical YBE we shall have a *standard* one. On the other hand, two Lie bialgebras (g, δ) and (g, δ') are said to be equivalent if there exists an automorphism O of g such that $\delta' = (O \otimes O) \circ \delta \circ O^{-1}$.

Let us now consider the $gl(2)$ Lie algebra

$$[J_3, J_+] = 2J_+ \qquad [J_3, J_-] = -2J_- \qquad [J_+, J_-] = J_3 \qquad [I, \cdot\,] = 0. \tag{2.3}$$

Notice that $gl(2) = sl(2,\mathbb{R})\oplus u(1)$ where I is the central generator. The second-order Casimir is

$$\mathcal{C} = J_3^2 + 2J_+J_- + 2J_-J_+. \tag{2.4}$$

The most general cocommutator $\delta : gl(2) \to gl(2)\otimes gl(2)$ will be a linear combination (with real coefficients)

$$\delta(X_i) = f_i^{jk}\, X_j \wedge X_k \tag{2.5}$$

of skewsymmetric products of the generators X_l of $gl(2)$. Such a completely general cocommutator has to be computed by firstly imposing the cocycle condition. This leads to the following six-parameter $\{a_+, a_-, b_+, b_-, a, b\}$ (pre)cocommutator:

$$\begin{aligned}\delta(J_3) &= a_+ J_3 \wedge J_+ + a_- J_3 \wedge J_- + b_+ J_+ \wedge I + b_- J_- \wedge I \\ \delta(J_+) &= a J_3 \wedge J_+ - \tfrac{1}{2} b_- J_3 \wedge I + a_- J_+ \wedge J_- + b J_+ \wedge I \\ \delta(J_-) &= a J_3 \wedge J_- - \tfrac{1}{2} b_+ J_3 \wedge I - a_+ J_+ \wedge J_- - b J_- \wedge I \\ \delta(I) &= 0.\end{aligned} \tag{2.6}$$

Afterwards, Jacobi identities have to be imposed onto $\delta^* : gl(2)^* \otimes gl(2)^* \to gl(2)^*$ in order to guarantee that a Lie bracket is defined through this map. Thus we obtain the following set of equations:

$$a_+ b - b_+ a = 0 \qquad a_+ b_- + a_- b_+ = 0 \qquad a_- b + b_- a = 0. \tag{2.7}$$

The next step is to find out the Lie bialgebras defined by (2.6) and (2.7) that come from classical r-matrices. Let us consider an arbitrary skewsymmetric element of $gl(2) \wedge gl(2)$:

$$r = c_1 J_3 \wedge J_+ + c_2 J_3 \wedge J_- + c_3 J_3 \wedge I + c_4 J_+ \wedge I + c_5 J_- \wedge I + c_6 J_+ \wedge J_-. \tag{2.8}$$

The corresponding Schouten bracket reads

$$\begin{aligned}[[r, r]] = {} & ({c_6}^2 - 4c_1 c_2) J_3 \wedge J_+ \wedge J_- + (c_4 c_6 - 2c_1 c_3) J_3 \wedge J_+ \wedge I \\ & + (2c_3 c_2 + c_6 c_5) J_3 \wedge J_- \wedge I + 2(c_2 c_4 + c_1 c_5) J_+ \wedge J_- \wedge I\end{aligned} \tag{2.9}$$

and the modified classical YBE will be satisfied, provided

$$c_4 c_6 - 2c_1 c_3 = 0 \qquad 2c_3 c_2 + c_6 c_5 = 0 \qquad c_2 c_4 + c_1 c_5 = 0. \tag{2.10}$$

These equations map exactly onto the conditions (2.7) (obtained from the Jacobi identities) under the following identification of the parameters:

$$\begin{aligned} a_+ &= 2c_1 & a_- &= -2c_2 & b_+ &= 2c_4 \\ b_- &= -2c_5 & a &= -c_6 & b &= -2c_3. \end{aligned} \tag{2.11}$$

Therefore all Lie bialgebras associated with $gl(2)$ are coboundaries and the most general r-matrix (2.8) can be written in terms of the a and b parameters:

$$r = \tfrac{1}{2}(a_+ J_3 \wedge J_+ - a_- J_3 \wedge J_- - b J_3 \wedge I + b_+ J_+ \wedge I - b_- J_- \wedge I - 2a J_+ \wedge J_-). \tag{2.12}$$

Under these conditions, the Schouten bracket reduces to

$$[[r, r]] = (c_6^2 - 4c_1 c_2) J_3 \wedge J_+ \wedge J_- = (a^2 + a_+ a_-) J_3 \wedge J_+ \wedge J_- \tag{2.13}$$

so that it allows us to distinguish between standard ($a^2 + a_+ a_- \neq 0$) and non-standard ($a^2 + a_+ a_- = 0$) Lie bialgebras.

On the other hand, the only element $\eta \in gl(2) \otimes gl(2)$ that is $Ad^{\otimes 2}$-invariant is given by

$$\eta = \tau_1 (J_3 \otimes J_3 + 2J_- \otimes J_+ + 2J_+ \otimes J_-) + \tau_2 I \otimes I \tag{2.14}$$

where τ_1 and τ_2 are arbitrary parameters. Since $r' = r + \eta$ will generate the same bialgebra as r, the element η will relate non-skewsymmetric r-matrices to skewsymmetric ones.

Let us now solve equations (2.7) explicitly; we find three disjoint families:

- *Family* I_+:
  Standard: $\{a_+ \neq 0,\ a_-,\ b_+,\ b_- = -a_- b_+ / a_+,\ a,\ b = b_+ a / a_+\}$ and $a^2 + a_+ a_- \neq 0$.
  Non-standard: $\{a_+ \neq 0,\ a_- = -a^2 / a_+,\ b_+,\ b_- = b_+ a^2 / a_+^2,\ a,\ b = b_+ a / a_+\}$.
- *Family* I_-:
  Standard: $\{a_+ = 0,\ a_- \neq 0,\ b_+ = 0,\ b_-,\ a \neq 0,\ b = -b_- a / a_-\}$.
  Non-standard: $\{a_+ = 0,\ a_- \neq 0,\ b_+ = 0,\ b_-,\ a = 0,\ b = 0\}$.

- *Family II*:
  Standard: $\{a_+ = 0,\ a_- = 0,\ b_+ = 0,\ b_- = 0,\ a \neq 0,\ b\}$.
  Non-standard: $\{a_+ = 0,\ a_- = 0,\ b_+,\ b_-,\ a = 0,\ b\}$.

This classification can be simplified by taking into account the following automorphism of $gl(2)$:

$$J_+ \to J_- \qquad J_- \to J_+ \qquad J_3 \to -J_3 \qquad I \to I \tag{2.15}$$

which leaves the Lie brackets (2.3) invariant. This map can be implemented at a Lie bialgebra level onto (2.6), and it implies a transformation of the deformation parameters of the form

$$\begin{array}{lll} a_+ \to a_- & a_- \to a_+ & b_+ \to -b_- \\ b_- \to -b_+ & a \to -a & b \to -b. \end{array} \tag{2.16}$$

The Jacobi identities (2.7) and the classical r-matrix (2.12) are invariant under the automorphism defined by (2.15) and (2.16). Therefore, the family I_- is included within I_+ provided $a_- = 0$. Hence we shall consider only the two families I_+ and II, whose explicit cocommutators and r-matrices are written in table 1. Note that the central generator I always has a vanishing cocommutator.

Table 1. Explicit cocommutators and r-matrices of $gl(2)$ Lie bialgebras.

	Family I_+	
	Standard ($a_+ \neq 0, a_-, b_+, a$ and $a^2 + a_+a_- \neq 0$)	Non-standard ($a_+ \neq 0, b_+, a$)
r	$\frac{1}{2}\left(a_+J_3 \wedge J_+ - a_-J_3 \wedge J_- - \frac{b_+a}{a_+}J_3 \wedge I + b_+J_+ \wedge I + \frac{a_-b_+}{a_+}J_- \wedge I - 2aJ_+ \wedge J_-\right)$	$\frac{1}{2}\left(a_+J_3 \wedge J_+ + \frac{a^2}{a_+}J_3 \wedge J_- - \frac{b_+a}{a_+}J_3 \wedge I + b_+J_+ \wedge I - \frac{b_+a^2}{a_+^2}J_- \wedge I - 2aJ_+ \wedge J_-\right)$
$\delta(J_3)$	$-(a_+J_+ + a_-J_-) \wedge J_3 + b_+\left(J_+ - \frac{a_-}{a_+}J_-\right) \wedge I$	$-a_+\left(J_+ - \frac{a^2}{a_+^2}J_-\right) \wedge J_3 + b_+\left(J_+ + \frac{a^2}{a_+^2}J_-\right) \wedge I$
$\delta(J_+)$	$(aJ_3 - a_-J_-) \wedge J_+ + \frac{b_+}{a_+}\left(aJ_+ + \frac{a_-}{2}J_3\right) \wedge I$	$a\left(J_3 + \frac{a}{a_+}J_-\right) \wedge J_+ + \frac{b_+a}{a_+}\left(J_+ - \frac{a}{2a_+}J_3\right) \wedge I$
$\delta(J_-)$	$(aJ_3 - a_+J_+) \wedge J_- - \frac{b_+}{2}\left(J_3 + \frac{2a}{a_+}J_-\right) \wedge I$	$(aJ_3 - a_+J_+) \wedge J_- - \frac{b_+}{2}\left(J_3 + \frac{2a}{a_+}J_-\right) \wedge I$
$\delta(I)$	0	0
	Family II	
	Standard ($a \neq 0, b$)	Non-standard (b_+, b_-, b)
r	$-\frac{1}{2}bJ_3 \wedge I - aJ_+ \wedge J_-$	$-\frac{1}{2}(bJ_3 - b_+J_+ + b_-J_-) \wedge I$
$\delta(J_3)$	0	$(b_+J_+ + b_-J_-) \wedge I$
$\delta(J_+)$	$-aJ_+ \wedge J_3 + bJ_+ \wedge I$	$-(\frac{1}{2}b_-J_3 - bJ_+) \wedge I$
$\delta(J_-)$	$-aJ_- \wedge J_3 - bJ_- \wedge I$	$-(\frac{1}{2}b_+J_3 + bJ_-) \wedge I$
$\delta(I)$	0	0

2.1. GL(2) Poisson–Lie groups

It is well known [27] that when a Lie bialgebra (g, δ) is a coboundary one with classical r-matrix $r = \sum_{i,j} r^{ij} X_i \otimes X_j$, the Poisson–Lie bivector Λ linked to it is given by the so-called Sklyanin bracket

$$\Lambda = \sum_{i,j} r^{ij} (X_i^L \otimes X_j^L - X_i^R \otimes X_j^R) \tag{2.17}$$

where X_i^L and X_j^R are left- and right-invariant vector fields on the Lie group $G = \mathrm{Lie}(g)$. We have just found that all $gl(2)$ Lie bialgebras are coboundary ones; therefore, we can deduce their corresponding Poisson–Lie groups by means of the Sklyanin bracket (2.17) as follows.

The 2×2 fundamental representation D of the $gl(2)$ algebra (2.3) is

$$\begin{aligned} D(J_3) &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad D(I) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ D(J_+) &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad D(J_-) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \end{aligned} \tag{2.18}$$

By using this representation, a group element of $GL(2)$ can be written as

$$T = e^{\theta_- D(J_-)} e^{\theta D(I)} e^{\theta_3 D(J_3)} e^{\theta_+ D(J_+)} = \begin{pmatrix} e^{\theta+\theta_3} & e^{\theta+\theta_3}\theta_+ \\ e^{\theta+\theta_3}\theta_- & e^{\theta+\theta_3}\theta_-\theta_+ + e^{\theta-\theta_3} \end{pmatrix}. \tag{2.19}$$

Now, the left- and right-invariant $GL(2)$ vector fields can be obtained:

$$\begin{aligned} X^L_{J_3} &= \partial_{\theta_3} - 2\theta_+\partial_{\theta_+} \qquad X^L_I = \partial_\theta \\ X^L_{J_+} &= \partial_{\theta_+} \qquad X^L_{J_-} = \theta_+\partial_{\theta_3} - \theta_+^2\partial_{\theta_+} + e^{-2\theta_3}\partial_{\theta_-} \end{aligned} \tag{2.20}$$

$$\begin{aligned} X^R_{J_3} &= \partial_{\theta_3} - 2\theta_-\partial_{\theta_-} \qquad X^R_I = \partial_\theta \\ X^R_{J_+} &= \theta_-\partial_{\theta_3} - \theta_-^2\partial_{\theta_-} + e^{-2\theta_3}\partial_{\theta_+} \qquad X^R_{J_-} = \partial_{\theta_-}. \end{aligned} \tag{2.21}$$

By substituting (2.20), (2.21) and the classical r-matrix (2.12) in the Sklyanin bracket (2.17) we obtain the following Poisson–Lie brackets between the (local) coordinates $\{\theta_-, \theta_+, \theta, \theta_3\}$:

$$\begin{aligned} \{\theta_+, \theta_3\} &= -a\theta_+ + \tfrac{1}{2}a_-\theta_+^2 - \tfrac{1}{2}a_+(1 - e^{-2\theta_3}) \\ \{\theta_-, \theta_3\} &= -a\theta_- + \tfrac{1}{2}a_+\theta_-^2 - \tfrac{1}{2}a_-(1 - e^{-2\theta_3}) \\ \{\theta_+, \theta_-\} &= (a_-\theta_+ - a_+\theta_-)e^{-2\theta_3} \\ \{\theta_+, \theta\} &= b\theta_+ + \tfrac{1}{2}b_-\theta_+^2 + \tfrac{1}{2}b_+(1 - e^{-2\theta_3}) \\ \{\theta_-, \theta\} &= -b\theta_- + \tfrac{1}{2}b_+\theta_-^2 + \tfrac{1}{2}b_-(1 - e^{-2\theta_3}) \\ \{\theta_3, \theta\} &= -\tfrac{1}{2}(b_+\theta_- + b_-\theta_+). \end{aligned} \tag{2.22}$$

By imposing Jacobi identities onto (2.22), conditions (2.7) restricting the space of Lie bialgebras are recovered. On the other hand, from (2.22) and (2.7) the Poisson–Lie groups associated with the families of $gl(2)$ Lie bialgebras written in table 1 can immediately be obtained explicitly.

We recall that a classification of Poisson–Lie structures on the group $GL(2)$ was carried out by Kupershmidt in [16], where quantum group structures on $GL(2)$ were also analysed. The relationship between (2.22) and such a classification can be explored by writing the matrix T (2.19) as

$$T = \begin{pmatrix} A & B \\ C & D \end{pmatrix}. \tag{2.23}$$

Starting from the Poisson brackets (2.22), the quadratic Poisson brackets between $\{A, B, C, D\}$ can be obtained:

$$
\begin{aligned}
\{A, C\} &= (a+b)AC - \left(\frac{a_+ + b_+}{2}\right) C^2 + \left(\frac{a_- - b_-}{2}\right)(A^2 + BC - AD) \\
\{A, B\} &= (a-b)AB - \left(\frac{a_- + b_-}{2}\right) B^2 + \left(\frac{a_+ - b_+}{2}\right)(A^2 + BC - AD) \\
\{B, D\} &= (a+b)BD - \left(\frac{a_+ + b_+}{2}\right)(D^2 + BC - AD) + \left(\frac{a_- - b_-}{2}\right) B^2 \\
\{C, D\} &= (a-b)CD + \left(\frac{a_+ - b_+}{2}\right) C^2 - \left(\frac{a_- + b_-}{2}\right)(D^2 + BC - AD) \\
\{A, D\} &= 2aBC - \left(\frac{a_+ + b_+}{2}\right) CD + \left(\frac{a_+ - b_+}{2}\right) AC - \left(\frac{a_- + b_-}{2}\right) BD \\
&\quad + \left(\frac{a_- - b_-}{2}\right) AB \\
\{B, C\} &= 2bBC - \left(\frac{a_+ + b_+}{2}\right) CD - \left(\frac{a_+ - b_+}{2}\right) AC + \left(\frac{a_- + b_-}{2}\right) BD \\
&\quad + \left(\frac{a_- - b_-}{2}\right) AB.
\end{aligned}
\tag{2.24}
$$

Therefore the Poisson structures given in [16] can be completely embedded within (2.24), provided the following identification is imposed:

$$
\begin{aligned}
&r = a + b \qquad s = -\tfrac{1}{2}(a_+ + b_+) \qquad v = b - a \\
&u = b_+ - a_+ \qquad w = 2a_- = 2b_-.
\end{aligned}
\tag{2.25}
$$

Here, r, s, u, v and w are the parameters arising in Kupershmidt's classification and $\{A, B, C, D\}$ are the corresponding generators (note that we have used capital letters for the latter in order to avoid confusion with the $gl(2)$ Lie bialgebra parameters). From (2.25) we conclude that Lie bialgebras having $a_- \neq b_-$ have no counterpart in [16]. For the remaining cases, (2.25) gives a straightforward correspondence between the quantum algebras that will be obtained after quantization and the quantum $GL(2)$ groups described in [16].

3. Harmonic oscillator Lie bialgebras through contractions

The $gl(2)$ algebra is isomorphic to the relativistic oscillator algebra introduced in [6] and its natural non-relativistic limit is the harmonic oscillator algebra h_4. Both algebras are related by means of a generalized Inönü–Wigner contraction [28]. If we define

$$
A_+ = \varepsilon J_+ \qquad A_- = \varepsilon J_- \qquad N = \tfrac{1}{2}(J_3 + I) \qquad M = \varepsilon^2 I \tag{3.1}
$$

the limit $\varepsilon \to 0$ of the Lie brackets obtained from (2.3) yields the oscillator algebra h_4

$$
[N, A_+] = A_+ \qquad [N, A_-] = -A_- \qquad [A_-, A_+] = M \qquad [M, \cdot\,] = 0 \tag{3.2}
$$

and the parameter can be interpreted as $\varepsilon = 1/c$, where c is the speed of light.

In what follows we work out the contractions from the multiparameter $gl(2)$ bialgebras written in table 1 to multiparameter h_4 bialgebras. The Lie bialgebra contraction (LBC) approach was introduced in [25] for a single deformation parameter. In order to perform an LBC we need two maps: the Lie algebra transformation (an Inönü–Wigner contraction as (3.1)), together with a mapping on the initial deformation parameter a:

$$a = \varepsilon^n a' \tag{3.3}$$

where n is any real number and a' is the contracted deformation parameter. The convergence of the classical r-matrix and the cocommutator δ under the limit $\varepsilon \to 0$ have to be analysed separately, since starting from a coboundary bialgebra, the LBC can lead to another coboundary bialgebra (both r and δ converge) or can produce a non-coboundary bialgebra (r diverges but δ converges). In other words, we have to find out the minimal value of the number n such that r converges, the minimal value of n such that δ converges, and finally to compare both of them [25].

In what follows, we show that the LBC method can be applied to multiparameter Lie bialgebras by considering a different map (3.3) for each deformation parameter. Let us describe this procedure by contracting, for instance, the non-standard family II given in table 1.

First, we analyse the classical r-matrix. We consider the following maps:

$$b_+ = 2\varepsilon^{n_+}\beta_+ \qquad b_- = -2\varepsilon^{n_-}\beta_- \qquad b = -\varepsilon^n \vartheta \tag{3.4}$$

where β_+, β_-, ϑ are the contracted deformation parameters, and n_+, n_-, n are real numbers to be determined by imposing the convergence of r under the non-relativistic limit. We introduce the Lie algebra contraction (3.1) and the maps (3.4) in the non-standard classical r-matrix of family II:

$$\begin{aligned} r &= -\tfrac{1}{2}(bJ_3 - b_+J_+ + b_-J_-) \wedge I \\ &= \tfrac{1}{2}\left(\varepsilon^n\vartheta(2N - M\varepsilon^{-2}) + 2\varepsilon^{n_+}\beta_+A_+\varepsilon^{-1} + 2\varepsilon^{n_-}\beta_-A_-\varepsilon^{-1}\right) \wedge M\varepsilon^{-2} \\ &= \left(\varepsilon^{n-2}\vartheta N + \varepsilon^{n_+-3}\beta_+A_+ + \varepsilon^{n_--3}\beta_-A_-\right) \wedge M. \end{aligned} \tag{3.5}$$

Thus the minimal values of n, n_+, n_- which allow r to converge under the limit $\varepsilon \to 0$ are given by

$$n = 2 \qquad n_+ = 3 \qquad n_- = 3 \tag{3.6}$$

and the contracted r-matrix turns out to be

$$r = (\vartheta N + \beta_+A_+ + \beta_-A_-) \wedge M. \tag{3.7}$$

Likewise we analyse the convergence of δ:

$$\begin{aligned} \delta(N) &= \tfrac{1}{2}\left(\delta(J_3) + \delta(I)\right) = \tfrac{1}{2}(b_+J_+ + b_-J_-) \wedge I \\ &= \tfrac{1}{2}\left(2\varepsilon^{n_+}\beta_+A_+\varepsilon^{-1} - 2\varepsilon^{n_-}\beta_-A_-\varepsilon^{-1}\right) \wedge M\varepsilon^{-2} \\ &= \left(\varepsilon^{n_+-3}\beta_+A_+ - \varepsilon^{n_--3}\beta_-A_-\right) \wedge M \end{aligned} \tag{3.8}$$

$$\begin{aligned} \delta(A_+) &= \varepsilon\delta(J_+) = -\varepsilon(\tfrac{1}{2}b_-J_3 - bJ_+) \wedge I \\ &= \varepsilon\left(\varepsilon^{n_-}\beta_-(2N - M\varepsilon^{-2}) - \varepsilon^n\vartheta A_+\varepsilon^{-1}\right) \wedge M\varepsilon^{-2} \\ &= \left(2\varepsilon^{n_--1}\beta_-N - \varepsilon^{n-2}\vartheta A_+\right) \wedge M \end{aligned} \tag{3.9}$$

$$\begin{aligned} \delta(A_-) &= \varepsilon\delta(J_-) = -\varepsilon(\tfrac{1}{2}b_+J_3 + bJ_-) \wedge I \\ &= -\varepsilon\left(\varepsilon^{n_+}\beta_+(2N - M\varepsilon^{-2}) - \varepsilon^n\vartheta A_-\varepsilon^{-1}\right) \wedge M\varepsilon^{-2} \\ &= -\left(2\varepsilon^{n_+-1}\beta_+N - \varepsilon^{n-2}\vartheta A_-\right) \wedge M \end{aligned} \tag{3.10}$$

and, obviously, $\delta(M) = 0$. Hence the minimal values of n, n_+, n_- which ensure the convergence of δ under the limit $\varepsilon \to 0$ read

$$n = 2 \qquad n_+ = 3 \qquad n_- = 3 \tag{3.11}$$

and the contracted cocommutator reduces to

$$\begin{aligned} &\delta(N) = (\beta_+ A_+ - \beta_- A_-) \wedge M \qquad && \delta(M) = 0 \\ &\delta(A_+) = -\vartheta A_+ \wedge M && \delta(A_-) = \vartheta A_- \wedge M. \end{aligned} \tag{3.12}$$

Therefore, in this case, the resulting contracted bialgebra is a coboundary one, as the contraction exponents coming from (3.6) and (3.11) coincide.

The remaining $gl(2)$ families of bialgebras can be contracted by following the LBC approach, and all the resulting contracted bialgebras are coboundaries. The transformations of the deformation parameters for the LBCs of the families I_+ and II read

I_+ Standard:	$a_+ = \varepsilon\alpha_+$	$a_- = -\varepsilon^3\beta_-$	$b_+ = -\varepsilon\alpha_+$	$a = \varepsilon^2\vartheta$
I_+ Non-standard:	$a_+ = \varepsilon\alpha_+$	$b_+ = -\varepsilon\alpha_+$	$a = \varepsilon^2\vartheta$	
II Standard:	$a = -\varepsilon^2\xi$	$b = -\varepsilon^2\vartheta$		
II Non-standard:	$b_+ = 2\varepsilon^3\beta_+$	$b_- = -2\varepsilon^3\beta_-$	$b = -\varepsilon^2\vartheta$	

If we apply these maps together with (3.1) to the $gl(2)$ Lie bialgebras displayed in table 1 and we take the limit $\varepsilon \to 0$, then the oscillator Lie bialgebras given in table 2 are derived. We stress that the LBC procedure just described can be applied in a similar way to any arbitrary multiparametric Lie bialgebra.

Table 2. Harmonic oscillator h_4 bialgebras via contraction from $gl(2)$.

	Family I_+	
	Standard ($\alpha_+ \neq 0$, ϑ, β_- and $\vartheta^2 - \alpha_+\beta_- \neq 0$)	Non-standard ($\alpha_+ \neq 0$, ϑ)
r	$\alpha_+ N \wedge A_+ + \vartheta(N \wedge M - A_+ \wedge A_-)$ $+\beta_- A_- \wedge M$	$\alpha_+ N \wedge A_+ + \vartheta(N \wedge M - A_+ \wedge A_-)$ $+(\vartheta^2/\alpha_+)A_- \wedge M$
$\delta(N)$	$\alpha_+ N \wedge A_+ - \beta_- A_- \wedge M$	$\alpha_+ N \wedge A_+ - (\vartheta^2/\alpha_+)A_- \wedge M$
$\delta(A_+)$	0	0
$\delta(A_-)$	$\alpha_+(N \wedge M - A_+ \wedge A_-) + 2\vartheta A_- \wedge M$	$\alpha_+(N \wedge M - A_+ \wedge A_-) + 2\vartheta A_- \wedge M$
$\delta(M)$	0	0
	Family II	
	Standard ($\xi \neq 0$, ϑ)	Non-standard (ϑ, β_+, β_-)
r	$\vartheta N \wedge M + \xi A_+ \wedge A_-$	$(\vartheta N + \beta_+ A_+ + \beta_- A_-) \wedge M$
$\delta(N)$	0	$(\beta_+ A_+ - \beta_- A_-) \wedge M$
$\delta(A_+)$	$-(\vartheta + \xi)A_+ \wedge M$	$-\vartheta A_+ \wedge M$
$\delta(A_-)$	$(\vartheta - \xi)A_- \wedge M$	$\vartheta A_- \wedge M$
$\delta(M)$	0	0

We recall that all oscillator bialgebras are coboundary ones [29] and they were explicitly obtained in [30]. In particular:

- Family I_+ corresponds to type I_+ of [30], except for the presence of the parameter β_+. However, this parameter is superfluous: if we define a new generator as $N' = N + (\beta_+/\alpha_+)M$ we find that the commutation rules (3.2) are preserved and β_+ appears explicitly in table 2.

- The bialgebras of type I_- of [30] are completely equivalent to those of type I_+ by means of an automorphism similar to that defined by (2.15) and (2.16) for $gl(2)$.
- The non-standard family II corresponds exactly to the non-standard type II of [30] but the standard subfamily does not, i.e. the parameters β_+ and β_- do not appear in the contracted bialgebras. We can introduce them by means of another automorphism defined through:

$$\begin{aligned} &N' = N - \frac{\beta_+}{\vartheta + \xi} A_+ - \frac{\beta_-}{\vartheta - \xi} A_- \qquad \vartheta + \xi \neq 0 \qquad \vartheta - \xi \neq 0 \\ &A'_+ = A_+ - \frac{\beta_-}{\vartheta - \xi} M \qquad A'_- = A_- - \frac{\beta_+}{\vartheta + \xi} M \qquad M' = M. \end{aligned} \tag{3.13}$$

These new generators satisfy the commutation rules (3.2) and now the standard family II can be identified within the classification of [30]. In particular, the harmonic oscillator Lie bialgebra corresponding to [26] is recovered in the case $\vartheta = 0$ and $\xi = -z$. As a byproduct, we have shown that when $\vartheta + \xi \neq 0$ and $\vartheta - \xi \neq 0$, both parameters β_+, β_- are irrelevant.

Therefore, there exist only two isolated oscillator Lie bialgebras that we do not find by contracting $gl(2)$: if $\vartheta = \xi$ it does not seem possible to introduce β_-, and likewise, if $\vartheta = -\xi$ to recover β_+. In the rest of the cases, the non-relativistic counterparts of $gl(2)$ algebraic structures can easily be obtained. In particular, Lie bialgebra contractions would give rise to (multiparametric) quantum h_4 algebras when applied onto the quantum $gl(2)$ deformations that will be considered in the following section.

4. Multiparametric quantum *gl*(2) algebras

Now we proceed to obtain some relevant quantum Hopf algebras corresponding to the $gl(2)$ bialgebras. We shall write only the coproducts and the deformed commutation rules, as the counit is always trivial and the antipode can easily be deduced by means of the Hopf algebra axioms. We emphasize that coproducts are found by computing a certain ‘exponential’ of the Lie bialgebra structure that characterizes the first order in the deformation. Deformed Casimir operators, which are essential for the construction of integrable systems, are also given explicitly.

4.1. Family I_+ quantizations

4.1.1. Standard subfamily with $a_- = 0$ and $b_+ = 0$. If a_- and b_+ vanish, we have $a_+ \neq 0$ and $a \neq 0$. Performing the following change of basis:

$$J'_3 = J_3 - \frac{a_+}{a} J_+ \tag{4.1}$$

the cocommutator adopts a simpler form

$$\delta(J'_3) = 0 \qquad \delta(J_+) = a J'_3 \wedge J_+ \qquad \delta(J_-) = a J'_3 \wedge J_- \qquad \delta(I) = 0 \tag{4.2}$$

while the classical r-matrix is formally preserved as $r = \frac{1}{2}(a_+ J'_3 \wedge J_+ - 2a J_+ \wedge J_-)$.

In this new $gl(2)$ basis the commutators (2.3) and Casimir (2.4) turn out to be

$$\begin{aligned} &[J'_3, J_+] = 2J_+ \qquad [J'_3, J_-] = -2J_- - \frac{a_+}{a} J'_3 - \frac{a_+^2}{a^2} J_+ \\ &[J_+, J_-] = J'_3 + \frac{a_+}{a} J_+ \qquad [I, \cdot\,] = 0. \end{aligned} \tag{4.3}$$

$$\mathcal{C} = \left(J'_3 + \frac{a_+}{a} J_+\right)^2 + 2J_+J_- + 2J_-J_+. \tag{4.4}$$

The coproduct of the corresponding quantum algebra $U_{a_+,a}(gl(2))$ can easily be deduced from (4.2) and reads

$$\begin{aligned} \Delta(J_3') &= 1\otimes J_3' + J_3'\otimes 1 \qquad & \Delta(J_+) &= e^{aJ_3'/2}\otimes J_+ + J_+\otimes e^{-aJ_3'/2} \\ \Delta(I) &= 1\otimes I + I\otimes 1 \qquad & \Delta(J_-) &= e^{aJ_3'/2}\otimes J_- + J_-\otimes e^{-aJ_3'/2}. \end{aligned} \tag{4.5}$$

We can return to the initial basis with J_3 instead of J_3'; however, in this case it does not seem worthy since J_3 and J_+ do not commute and this fact would complicate further computations (note also that J_3' is primitive so that we know that $\Delta(e^{xJ_3'}) = e^{xJ_3'}\otimes e^{xJ_3'}$ for any parameter x).

The deformed commutation rules compatible with (4.5) are found to be

$$\begin{aligned} &[J_3', J_+] = 2J_+ \qquad [J_3', J_-] = -2J_- - \frac{a_+}{a}\frac{\sinh(aJ_3'/2)}{a/2} - \frac{a_+^2}{a^2}J_+ \qquad [I,\cdot\,] = 0 \\ &[J_+, J_-] = \frac{\sinh aJ_3'}{a} + \frac{a_+}{a}\left(\frac{e^a-1}{2a}\right)\left(e^{-aJ_3'/2}J_+ + J_+e^{aJ_3'/2}\right) \end{aligned} \tag{4.6}$$

and the central element that deforms the Casimir (4.4) is

$$\begin{aligned} \mathcal{C} = {}& \frac{2}{a\tanh a}\left(\cosh(aJ_3')-1\right) + \frac{a_+}{a}\left(\frac{\sinh(aJ_3'/2)}{a/2}J_+ + J_+\frac{\sinh(aJ_3'/2)}{a/2}\right) \\ & + \frac{a_+^2}{a^2}J_+^2 + 2(J_+J_- + J_-J_+). \end{aligned} \tag{4.7}$$

In order to check these results, the following relations are useful:

$$\begin{aligned} &e^{xJ_3'}J_-e^{-xJ_3'} = J_-e^{-2x} + \frac{a_+}{a^2}(e^{-2x}-1)\sinh(aJ_3'/2) - J_+\frac{a_+^2}{2a^2}\sinh 2x \\ &e^{xJ_3'}J_+e^{-xJ_3'} = J_+e^{2x}. \end{aligned} \tag{4.8}$$

We remark that the $u(1)$ (central) generator I here does not couple with the $sl(2,\mathbb{R})$ sector, so $U_{a_+,a}(gl(2)) = U_{a_+,a}(sl(2,\mathbb{R}))\oplus u(1)$.

It is also interesting to stress that the (to our knowledge, new) quantum algebra $U_{a_+,a}(sl(2,\mathbb{R}))$ is just a superposition of the standard and non-standard deformations of $sl(2,\mathbb{R})$, since its classical r-matrix is the sum of both the standard and the non-standard one for $sl(2,\mathbb{R})$. This fact can be clearly appreciated by deducing the associated quantum R-matrix in the fundamental representation. By following [31], we get a 2×2 matrix representation D of (4.6) given by

$$\begin{aligned} &D(J_3') = \begin{pmatrix} 1 & -a_+/a \\ 0 & -1 \end{pmatrix} \qquad D(J_+) = \begin{pmatrix} 0 & \cosh(\frac{1}{2}a) \\ 0 & 0 \end{pmatrix} \qquad D(I) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &D(J_-) = \begin{pmatrix} 0 & (a_+^2/4a^2)\left((2/a)\sinh(\frac{1}{2}a) - \cosh(\frac{1}{2}a)\right) \\ (2/a)\sinh(\frac{1}{2}a) & 0 \end{pmatrix} \end{aligned} \tag{4.9}$$

which, in turn, provides a 4×4 matrix representation of the coproduct (4.5). We consider now an arbitrary 4×4 matrix and impose it to fulfil both the quantum YBE and the property

$$\mathcal{R}\Delta(X)\mathcal{R}^{-1} = \sigma\circ\Delta(X) \tag{4.10}$$

for $X\in\{D(J_3'), D(J_+), D(J_-), D(I)\}$, and where $\sigma(A\otimes B) = B\otimes A$. Finally we find the solution

$$\mathcal{R} = \begin{pmatrix} 1 & h & -qh & h^2 \\ 0 & q & 1-q^2 & qh \\ 0 & 0 & q & -h \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{4.11}$$

where

$$q = e^{a} \qquad h = \frac{a_+}{2}\left(\frac{e^{a}-1}{a}\right). \tag{4.12}$$

The expression (4.11) clearly shows the intertwining between the standard and non-standard properties within the quantum algebra $U_{a_+,a}(gl(2))$. This results in $\mathcal{R}$ being a quasitriangular solution of the quantum YBE and not a triangular one, since $\mathcal{R}_{12}\mathcal{R}_{21} \neq I$. In the fundamental representation (4.9), the standard quantum R-matrix of $sl(2,\mathbb{R})$ would be obtained in the limit $a_+ \to 0$, and the non-standard or Jordanian one [13, 14] would be a consequence of taking $a \to 0$. However, we stress that the latter is not a well-defined limit at the Hopf algebra level (see [32] for a detailed study of this kind of problem). Moreover, $U_{a_+,a}(gl(2))$ is just the quantum algebra underlying the construction of non-standard quantum R-matrices out of standard ones proposed in [33, 34] and its dual Hopf algebra would give rise to the quantum group $GL_{h,q}(2)$ introduced in [16]. We finally recall that the classification of 4×4 constant solutions of the quantum YBE can be found in [35].

4.1.2. Non-standard subfamily with a = 0. We now restrict ourselves to the case with $a = 0$ so that J_+ is a primitive generator. The coproduct can easily be deduced by applying the Lyakhovsky–Mudrov method [36] in the same way as in the oscillator h_4 case [30]. The cocommutators for the two non-primitive generators can be written in matrix form as

$$\delta\begin{pmatrix} J_3' \\ J_- \end{pmatrix} = \begin{pmatrix} -a_+J_+ & 0 \\ \frac{1}{2}b_+I & -a_+J_+ \end{pmatrix} \dot{\wedge} \begin{pmatrix} J_3' \\ J_- \end{pmatrix} \tag{4.13}$$

where

$$J_3' := J_3 - \frac{b_+}{a_+}I \qquad a_+ \neq 0. \tag{4.14}$$

Hence their coproduct is given by

$$\Delta\begin{pmatrix} J_3' \\ J_- \end{pmatrix} = \begin{pmatrix} 1\otimes J_3' \\ 1\otimes J_- \end{pmatrix} + \sigma\left(\exp\left\{\begin{pmatrix} a_+J_+ & 0 \\ -\frac{1}{2}b_+I & a_+J_+ \end{pmatrix}\right\} \dot{\otimes} \begin{pmatrix} J_3' \\ J_- \end{pmatrix}\right) \tag{4.15}$$

where $\sigma(X\otimes Y) := Y \otimes X$. The exponential of the Lie bialgebra matrix coming from (4.13) is the essential object in the obtention of the deformed coproduct, whose coassociativity is ensured by construction [36]. In terms of the original basis the coproduct, commutation rules and Casimir of the quantum $gl(2)$ algebra, $U_{a_+,b_+}(gl(2))$, are given by

$$\begin{aligned} &\Delta(J_+) = 1\otimes J_+ + J_+\otimes 1 \qquad \Delta(I) = 1\otimes I + I\otimes 1 \\ &\Delta(J_3) = 1\otimes J_3 + J_3\otimes e^{a_+J_+} - b_+I\otimes\left(\frac{e^{a_+J_+}-1}{a_+}\right) \\ &\Delta(J_-) = 1\otimes J_- + J_-\otimes e^{a_+J_+} - \frac{b_+}{2}\left(J_3 - \frac{b_+}{a_+}I\right)\otimes I e^{a_+J_+} \end{aligned} \tag{4.16}$$

$$\begin{aligned} &[J_3, J_+] = 2\frac{e^{a_+J_+}-1}{a_+} \qquad [J_3, J_-] = -2J_- + \frac{a_+}{2}\left(J_3 - \frac{b_+}{a_+}I\right)^2 \\ &[J_+, J_-] = J_3 + b_+I\frac{e^{a_+J_+}-1}{a_+} \qquad [I, \cdot\,] = 0 \end{aligned} \tag{4.17}$$

$$\begin{aligned} \mathcal{C} = {} & \left(J_3 - \frac{b_+}{a_+}I\right)e^{-a_+J_+}\left(J_3 - \frac{b_+}{a_+}I\right) + 2\frac{b_+}{a_+}J_3I \\ & + 2\frac{1-e^{-a_+J_+}}{a_+}J_- + 2J_-\frac{1-e^{-a_+J_+}}{a_+} + 2(e^{-a_+J_+}-1). \end{aligned} \tag{4.18}$$

It is interesting to note that $U_{a_+,b_+}(gl(2))$ reproduces the two-parameter Jordanian deformation of $gl(2)$ obtained in [24] once we relabel the deformation parameters as $a_+ = 2h$ and $b_+ = -2s$. We also remark that this quantum deformation was constructed in [22] by using a duality procedure from the quantum group $GL_{g,h}(2)$ introduced in [15]; it can be checked that the generators $\{A, B, H, Y\}$ and deformation parameters $\tilde{g}, \tilde{h}$ defined by

$$\begin{aligned} &A = I \qquad B = J_+ \\ &H = \exp\{-a_+ J_+/2\} J_3 + 2\frac{b_+}{a_+} I \sinh(a_+ J_+/2) \\ &Y = \exp\{-a_+ J_+/2\} J_- - \frac{b_+^2}{4a_+} \exp\{a_+ J_+/2\} I^2 + \frac{a_+}{8} \sinh(a_+ J_+/2) \\ &\tilde{g} = -a_+/2 \qquad \tilde{h} = -b_+/2 \end{aligned} \tag{4.19}$$

give rise to the quantum $gl(2)$ algebra worked out in [22]. On the other hand, we also recover the quantum extended $sl(2,\mathbb{R})$ algebra introduced in [23] if we consider the basis $\{J_3', J_+, J_-, I\}$ and set $a_+ = 2z$ and $b_+ = -2z$. The corresponding universal R-matrix can be also found in [23, 24].

In the basis adopted here, a coupling of the central generator I with the $sl(2,\mathbb{R})$ sector arises; however, if we set $b_+ = 0$ this coupling disappears and we can rewrite $U_{a_+}(gl(2)) = U_{a_+}(sl(2,\mathbb{R})) \oplus u(1)$ where $U_{a_+}(sl(2,\mathbb{R}))$ is the non-standard or Jordanian deformation of $sl(2,\mathbb{R})$ [13, 14, 37–40].

4.2. Family II quantizations

4.2.1. Standard subfamily and twisted XXZ models. The coproduct and commutators of the two-parametric quantum algebra $U_{a,b}(gl(2))$ are given by

$$\begin{aligned} &\Delta(I) = 1 \otimes I + I \otimes 1 \qquad \Delta(J_3) = 1 \otimes J_3 + J_3 \otimes 1 \\ &\Delta(J_+) = e^{(aJ_3 - bI)/2} \otimes J_+ + J_+ \otimes e^{-(aJ_3 - bI)/2} \\ &\Delta(J_-) = e^{(aJ_3 + bI)/2} \otimes J_- + J_- \otimes e^{-(aJ_3 + bI)/2} \end{aligned} \tag{4.20}$$

$$[J_3, J_+] = 2J_+ \qquad [J_3, J_-] = -2J_- \qquad [J_+, J_-] = \frac{\sinh aJ_3}{a} \qquad [I, \cdot\,] = 0. \tag{4.21}$$

The deformed Casimir is

$$\mathcal{C} = \cosh a \left(\frac{\sinh(aJ_3/2)}{a/2} \right)^2 + 2\,\frac{\sinh a}{a}\,(J_+ J_- + J_- J_+). \tag{4.22}$$

This quantum algebra, together with its corresponding universal quantum R-matrix, was obtained in [18] and [20], and it can be related to the so-called $gl_{q,s}(2)$ introduced in [17] (see also [19]) by defining a set of new generators in the form

$$\begin{aligned} &\widetilde{J}_0 = \frac{1}{2} J_3 \qquad \widetilde{J}_+ = \sqrt{\frac{a}{\sinh a}} \exp\left\{ \frac{b}{2a} I \right\} J_+ \\ &\widetilde{Z} = \frac{b}{2a} I \qquad \widetilde{J}_- = \sqrt{\frac{a}{\sinh a}} \exp\left\{ -\frac{b}{2a} I \right\} J_- \end{aligned} \tag{4.23}$$

and the parameters q and s as

$$q = e^{\eta} \qquad \eta = -a \qquad s^{\tilde{Z}} = \exp\left\{ \frac{b}{2a} I \right\}. \tag{4.24}$$

The algebra $U_{a,b}(gl(2))$ is just the quantum algebra underlying the XXZ Heisenberg Hamiltonian with twisted boundary conditions [1]. This deformation can be thought as a Reshetikhin twist of the usual standard deformation. This superposition of the standard quantization and a twist is easily reflected at the Lie bialgebra level by the associated classical r-matrix $r = -\frac{1}{2}bJ_3 \wedge I - aJ_+ \wedge J_-$ (see table 1): within it, the second term generates the standard deformation and the exponential of the first one gives us the Reshetikhin twist. Compatibility between both quantizations is ensured by the fact that r fulfils the modified classical YBE, and the method used here shows that the full simultaneous quantization of the two-parameter Lie bialgebra is possible.

On the other hand, if we set $b = 0$ we find that I does not couple with the deformation of the $sl(2,\mathbb{R})$ sector, and $U_a(gl(2)) = U_a(sl(2,\mathbb{R})) \oplus u(1)$ where $U_a(sl(2,\mathbb{R}))$ is the well known standard deformation of $sl(2,\mathbb{R})$ [10, 41].

4.2.2. Non-standard subfamily and twisted XXX models. This bialgebra has one primitive generator I; the cocommutator for the remaining generators can be written as

$$\delta \begin{pmatrix} J_3 \\ J_+ \\ J_- \end{pmatrix} = \begin{pmatrix} 0 & -b_+I & -b_-I \\ \frac{1}{2}b_-I & -bI & 0 \\ \frac{1}{2}b_+I & 0 & bI \end{pmatrix} \dot{\wedge} \begin{pmatrix} J_3 \\ J_+ \\ J_- \end{pmatrix} \tag{4.25}$$

so that their coproduct is given by

$$\Delta \begin{pmatrix} J_3 \\ J_+ \\ J_- \end{pmatrix} = \begin{pmatrix} 1\otimes J_3 \\ 1\otimes J_+ \\ 1\otimes J_- \end{pmatrix} + \sigma \left(\exp\left\{ \begin{pmatrix} 0 & b_+I & b_-I \\ -\frac{1}{2}b_-I & bI & 0 \\ -\frac{1}{2}b_+I & 0 & -bI \end{pmatrix} \right\} \dot{\otimes} \begin{pmatrix} J_3 \\ J_+ \\ J_- \end{pmatrix} \right). \tag{4.26}$$

If we denote the exponential of the Lie bialgebra matrix by E, the coproduct can be expressed in terms of the E_{ij} entries as follows:

$$\begin{aligned} &\Delta(I) = 1\otimes I + I \otimes 1 \\ &\Delta(J_3) = 1\otimes J_3 + J_3 \otimes E_{11}(I) + J_+ \otimes E_{12}(I) + J_- \otimes E_{13}(I) \\ &\Delta(J_+) = 1\otimes J_+ + J_+ \otimes E_{22}(I) + J_3 \otimes E_{21}(I) + J_- \otimes E_{23}(I) \\ &\Delta(J_-) = 1\otimes J_- + J_- \otimes E_{33}(I) + J_3 \otimes E_{31}(I) + J_+ \otimes E_{32}(I). \end{aligned} \tag{4.27}$$

The explicit form of the functions E_{ij} is quite complicated, which in turn makes it difficult to find the associated deformed commutation relations. Therefore in what follows we study a specific case by setting $b_- = 0$. The coproduct of the quantum algebra $U_{b_+,b}(gl(2))$ is

$$\begin{aligned} &\Delta(I) = 1\otimes I + I\otimes 1 \qquad \Delta(J_+) = 1\otimes J_+ + J_+ \otimes e^{bI} \\ &\Delta(J_3) = 1\otimes J_3 + J_3\otimes 1 + b_+J_+ \otimes \left(\frac{e^{bI}-1}{b}\right) \\ &\Delta(J_-) = 1\otimes J_- + J_-\otimes e^{-bI} + b_+J_3 \otimes \left(\frac{e^{-bI}-1}{2b}\right) + b_+^2 J_+ \otimes \left(\frac{1-\cosh bI}{2b^2}\right) \end{aligned} \tag{4.28}$$

and the associated commutation rules are the non-deformed ones (2.3). The role of I is essential in this deformation, and no uncoupled structure can be recovered unless all deformation parameters vanish. On the other hand, the element

$$\mathcal{R} = \exp\{r\} = \exp\{I\otimes(bJ_3 - b_+J_+)/2\}\exp\{-(bJ_3 - b_+J_+)\otimes I/2\} \tag{4.29}$$

is a solution of the quantum YBE (as I is a central generator) and it also fulfils relation (4.10). The proof of this property is sketched in the appendix. In the fundamental representation (2.18), the R-matrix (4.29) reads

$$\mathcal{R} = \begin{pmatrix} 1 & -e^{-b}\,p & p & -e^{-b}\,p^2 \\ 0 & e^{-b} & 0 & e^{-b}\,p \\ 0 & 0 & e^{b} & -p \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad p = \frac{b_+}{2}\left(\frac{e^b - 1}{b}\right). \tag{4.30}$$

From the point of view of spin systems, a direct connection can be established between the one-parameter deformation with $b_+ = b_- = 0$ and the twisted XXX chain. This particular quantization can be obtained as the limit $a \to 0$ of the (standard) quantum algebra $U_{a,b}(gl(2))$, and a is known to be related to the anisotropy of the XXZ model. Under such a limit, twisted boundary conditions coming from $b \neq 0$ are preserved, and a twisted XXX model is expected to arise. This symmetry property can be checked explicitly by following the approach presented in [42] (and used there in order to obtain deformed t–J models). We consider the fundamental representation D of $U_b(gl(2))$ (2.18) in terms of Pauli spin matrices: $D(J_3) = \sigma_3$, $D(J_+) = \sigma_+$, $D(J_-) = \sigma_-$ and $D(I)$ is again the two-dimensional identity matrix. If we compute (with $b_+ = 0$) the deformed coproduct (4.28) of the Casimir (2.4), we obtain

$$(D \otimes D)(\Delta_b(\mathcal{C})) = 6 + 2\,(\sigma_3 \otimes \sigma_3 + 2\,e^{-b}\,\sigma_- \otimes \sigma_+ + 2\,e^{b}\sigma_+ \otimes \sigma_-). \tag{4.31}$$

This means that the twisted XXX Heisenberg Hamiltonian can be written (up to global constants) as the sum of elementary two-site Hamiltonians given by the coproducts (4.31):

$$\begin{aligned} H_b &= \sum_{i=1}^{N} (D_i \otimes D_{i+1})(\Delta_b^{i,i+1}(\mathcal{C})) \\ &= 6\,N + 2\,\sum_{i=1}^{N} (\sigma_3^i \sigma_3^{i+1} + 2\,e^{-b}\,\sigma_-^i \sigma_+^{i+1} + 2\,e^{b}\sigma_+^i \sigma_-^{i+1}). \end{aligned} \tag{4.32}$$

This expression explicitly reflects the $U_b(gl(2))$ quantum algebra invariance of this model since, by construction, the Hamiltonian (4.32) commutes with the $(N+1)$th coproduct of the generators of $U_b(gl(2))$. In the same way, further contributions could be obtained by considering other quantum deformations belonging to this family. In particular, if we take the two-parametric coproduct (4.28) and repeat the same construction we are led to the following spin Hamiltonian:

$$\begin{aligned} H_{b_+,b} &= \sum_{i=1}^{N} (D_i \otimes D_{i+1})(\Delta_{b_+,b}^{i,i+1}(\mathcal{C})) \\ &= 6\,N + 2\,\sum_{i=1}^{N} \left(\sigma_3^i \sigma_3^{i+1} + 2\,e^{-b}\,\sigma_-^i \sigma_+^{i+1} + 2\,e^{b}\sigma_+^i \sigma_-^{i+1}\right) \\ &\quad + 2b_+ \sum_{i=1}^{N} \left\{ \left(\frac{e^b - 1}{b}\right) \sigma_+^i \sigma_3^{i+1} + \left(\frac{e^{-b} - 1}{b}\right) \sigma_3^i \sigma_+^{i+1} \right\} \\ &\quad + 2b_+^2 \sum_{i=1}^{N} \left(\frac{1 - \cosh b}{b^2}\right) \sigma_+^i \sigma_+^{i+1}. \end{aligned} \tag{4.33}$$

Therefore, we have obtained a (quadratic in b_+) deformation of the twisted XXX chain, which is invariant under $U_{b_+,b}(gl(2))$ and whose associated quantum R-matrix is (4.30). Likewise, the introduction of the full quantization containing b_- provides a further deformation of the Hamiltonian (4.33).

5. Concluding remarks

We have presented a constructive overview of multiparameter quantum $gl(2)$ deformations based on the classification and further quantization of $gl(2)$ Lie bialgebra structures. The quantization procedure (based on the construction of the 'exponential' to first order in the deformation) turns out to be extremely efficient for constructing multiparametric quantum $gl(2)$ algebras explicitly. By following this method, a family of new multiparametric quantizations generalizing the symmetries of twisted XXX models is introduced, and the quantum algebra counterpart of the superposition of standard and non-standard deformations $GL_{h,q}(2)$ [16] is obtained. Throughout the paper, Lie bialgebra analysis is shown to provide essential algebraic information characterizing the quantum algebras and their associated models. For instance, a Lie bialgebra contraction method gives a straightforward method for implementing the non-relativistic limit of the quantum $gl(2)$ algebras, different coupling possibilities between the central generator and the $sl(2,\mathbb{R})$ substructure are easily extracted from the cocommutator δ, and Reshetikhin twists giving rise to twisted XXZ models can be identified (and explicitly constructed) with the help of the classical r-matrices generating the Lie bialgebras. In general, we can conclude that the existence of a central generator strongly increases the number of different quantizations, even when this central extension is cohomologically trivial at the non-deformed level (compare the classification here presented with the one corresponding to $sl(2,\mathbb{R})$), and the explicit construction of these quantizations provide an algebraic background for systematically obtaining new integrable systems.

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Appendix

We prove here that the element (4.29) satisfies (4.10) for the generator J_-. We write the universal R-matrix as $\mathcal{R} = \exp\{I \otimes A\} \exp\{-A \otimes I\}$ where $A = \frac{1}{2}(bJ_3 - b_+ J_+)$ and we take into account the formula

$$e^f\,\Delta(X)\,e^{-f} = \Delta(X) + \sum_{n=1}^{\infty} \frac{1}{n!}\,[f, \ldots [f, \Delta(X)]^{n)} \ldots]. \tag{A.1}$$

We set $f \equiv -A \otimes I$ and we consider the coproduct of J_- (4.28); thus we obtain

$$\begin{aligned}
[f, \ldots [f, \Delta(J_-)]^{n)} \ldots] &= \left(J_- + \frac{b_+}{2b} J_3\right) \otimes (bI)^n e^{-bI} \\
&\quad + \frac{b_+^2}{2b^2} J_+ \otimes (bI)^n \sinh bI \qquad \text{for } n \text{ odd and } n \geqslant 1 \\
[f, \ldots [f, \Delta(J_-)]^{n)} \ldots] &= \left(J_- + \frac{b_+}{2b} J_3\right) \otimes (bI)^n e^{-bI} \\
&\quad - \frac{b_+^2}{2b^2} J_+ \otimes (bI)^n \cosh bI \qquad \text{for } n \text{ even and } n \geqslant 2.
\end{aligned} \tag{A.2}$$

Therefore

$$
\begin{aligned}
e^f \Delta(J_-)e^{-f} &= \Delta(J_-) + (J_- + \frac{b_+}{2b} J_3) \otimes e^{-bI} \sum_{n=1}^{\infty} \frac{(bI)^n}{n!} \\
&\quad + \frac{b_+^2}{2b^2} J_+ \otimes \sinh bI \sum_{k=0}^{\infty} \frac{(bI)^{2k+1}}{(2k+1)!} - \frac{b_+^2}{2b^2} J_+ \otimes \cosh bI \sum_{k=1}^{\infty} \frac{(bI)^{2k}}{(2k)!} \\
&= \Delta(J_-) + \left(J_- + \frac{b_+}{2b} J_3 \right) \otimes (1 - e^{-bI}) + \frac{b_+^2}{2b^2} J_+ \otimes (\cosh bI - 1) \\
&= 1 \otimes J_- + J_- \otimes 1 \equiv \Delta_0(J_-).
\end{aligned}
\tag{A.3}
$$

Now we take $f \equiv I \otimes A$ and we find that

$$
\begin{aligned}
&[f, \ldots [f, \Delta_0(J_-)]^{n)} \ldots] = -(bI)^n \otimes \left(J_- + \frac{b_+}{2b} J_3 \right) \qquad \text{for } n \text{ odd and } n \geqslant 1 \\
&[f, \ldots [f, \Delta_0(J_-)]^{n)} \ldots] = (bI)^n \otimes \left(J_- + \frac{b_+}{2b} J_3 - \frac{b_+^2}{2b^2} J_+ \right) \qquad \text{for } n \text{ even and } n \geqslant 2.
\end{aligned}
\tag{A.4}
$$

Finally, the proof follows from

$$
\begin{aligned}
e^f \Delta_0(J_-)e^{-f} &= 1 \otimes J_- + J_- \otimes 1 - \sinh bI \otimes \left(J_- + \frac{b_+}{2b} J_3 \right) \\
&\quad + (\cosh bI - 1) \otimes \left(J_- + \frac{b_+}{2b} J_3 - \frac{b_+^2}{2b^2} J_+ \right) = \sigma \circ \Delta(J_-).
\end{aligned}
\tag{A.5}
$$

Likewise, it can be checked that (4.10) is fulfilled for the remaining generators.

References

[1] Monteiro M R, Roditi I, Rodrigues L M C S and Sciuto S 1995 *Phys. Lett.* B **354** 389
Monteiro M R, Roditi I, Rodrigues L M C S and Sciuto S 1995 *Mod. Phys. Lett.* A **10** 419

[2] Alcaraz F, Grimm U and Rittenberg V 1989 *Nucl. Phys.* B **316** 735

[3] Pasquier V and Saleur H 1990 *Nucl. Phys.* B **330** 523

[4] Foerster A, Links J and Roditi I 1998 *J. Phys. A: Math. Gen.* **31** 687

[5] Reshetikhin N 1990 *Lett. Math. Phys* **20** 331

[6] Aldaya V, Bisquert J and Navarro-Salas J 1991 *Phys. Lett.* A **156** 381
Aldaya V and Guerrero J 1993 *J. Phys. A: Math. Gen.* **26** L1195

[7] Ballesteros A and Ragnisco O 1998 *J. Phys. A: Math. Gen.* **31** 3791

[8] Ballesteros A and Herranz F J 1998 Long range integrable oscillator chains from quantum algebras *Preprint* solv-int/9805004

[9] Ballesteros A, Herranz F J and Parashar P 1998 *Mod. Phys. Lett.* A **13** 1241

[10] Drinfeld V G 1986 *Quantum Groups (Proc. Int. Congress Math.)* (Berkeley, CA: MRSI) p 798

[11] Manin Yu I 1988 Quantum groups and non-commutative geometry *Preprint* CRM-1561, Montréal

[12] Reshetikhin N Yu, Takhtajan L A and Faddeev L D 1990 *Leningrad Math. J.* **1** 193

[13] Demidov E E, Manin Yu I, Mukhin E E and Zhdanovich D V 1990 *Progr. Theor. Phys. Suppl.* **102** 203

[14] Zakrzewski S 1991 *Lett. Math. Phys.* **22** 287

[15] Aghamohammadi A 1993 *Mod. Phys. Lett.* A **8** 2607

[16] Kupershmidt B A 1994 *J. Phys. A: Math. Gen.* **27** L47

[17] Schirrmacher A, Wess J and Zumino B 1991 *Z. Phys.* C **49** 317

[18] Dobrev V K 1992 *J. Math. Phys.* **33** 3419

[19] Burdík C and Hellinger P 1992 *J. Phys. A: Math. Gen.* **25** L629

[20] Chakrabarti R and Jagannathan R 1994 *J. Phys. A: Math. Gen.* **27** 2023

[21] Fronsdal C and Galindo A 1993 *Lett. Math. Phys.* **27** 59

[22] Aneva B L, Dobrev V K and Mihov S G 1997 *J. Phys. A: Math. Gen.* **30** 6769
[23] Ballesteros A, Herranz F J and Negro J 1997 *J. Phys. A: Math. Gen.* **30** 6797
[24] Parashar P 1998 *Lett. Math. Phys.* **45** 105
[25] Ballesteros A, Gromov N A, Herranz F J, del Olmo M A and Santander M 1995 *J. Math. Phys.* **36** 5916
[26] Gómez C and Sierra G 1993 *J. Math. Phys.* **34** 2119
[27] Drinfel'd V G 1983 *Sov. Math. Dokl.* **27** 68
[28] Weimar-Woods E 1995 *J. Math. Phys.* **36** 4519
[29] Ballesteros A and Herranz F J 1997 Harmonic oscillator Lie bialgebras and their quantization *Quantum Group Symposium at Group21* ed H D Doebner and V K Dobrev (Sofia: Heron) p 379
[30] Ballesteros A and Herranz F J 1996 *J. Phys. A: Math. Gen.* **29** 4307
[31] Ballesteros A, Celeghini E, Giachetti R, Sorace E and Tarlini M 1993 *J. Phys. A: Math. Gen.* **26** 7495
[32] Kulish P P, Lyakhovsky V D and Mudrov A I 1998 Extended jordanian twists for Lie algebras *Preprint* math.QA/9806014
[33] Aghamohammadi A, Khorrami M and Shariati A 1995 *J. Phys. A: Math. Gen.* **28** L225
[34] Abdesselam B, Chakrabarti A and Chakrabarti R 1998 *Mod. Phys. Lett.* A to appear
(Abdesselam B, Chakrabarti A and Chakrabarti R 1997 *Preprint* q-alg/9706033)
[35] Hlavaty L 1992 *J. Phys. A: Math. Gen.* **25** L63
Hietarinta J 1992 *Phys. Lett.* A **165** 245
[36] Lyakhovsky V and Mudrov A 1992 *J. Phys. A: Math. Gen.* **25** L1139
[37] Ohn C 1992 *Lett. Math. Phys.* **25** 85
[38] Vladimirov A A 1993 *Mod. Phys. Lett.* A **8** 2573
[39] Shariati A, Aghamohammadi A and Khorrami M 1996 *Mod. Phys. Lett.* A **11** 187
[40] Ballesteros A and Herranz F J 1996 *J. Phys. A: Math. Gen.* **29** L311
[41] Kulish P P and Reshetikhin N Yu 1983 *J. Sov. Math.* **23** 2435
[42] Arnaudon D, Chryssomalakos C and Frappat L 1995 *J. Math. Phys.* **36** 5262